

Fermionic T-duality in fermionic double space *

B. Nikolić [†] and B. Sazdović [‡]

Institute of Physics

University of Belgrade

P.O.Box 57, 11001 Belgrade, Serbia

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Abstract

In this article we offer the interpretation of the fermionic T-duality of the type II superstring theory in double space. We generalize the idea of double space doubling the fermionic sector of the superspace. In such doubled space fermionic T-duality is represented as permutation of the fermionic coordinates θ^α and $\bar{\theta}^\alpha$ with the corresponding fermionic T-dual ones, ϑ_α and $\bar{\vartheta}_\alpha$, respectively. Demanding that T-dual transformation law has the same form as initial one, we obtain the known form of the fermionic T-dual NS-R i R-R background fields. Fermionic T-dual NS-NS background fields are obtained under some assumptions. We conclude that only symmetric part of R-R field strength and symmetric part of its fermionic T-dual contribute to the fermionic T-duality transformation of dilaton field and analyze the dilaton field in fermionic double space. As a model we use the ghost free action of type II superstring in pure spinor formulation in approximation of constant background fields up to the quadratic terms.

1 Introduction

Two theories T-dual to one another can be viewed as being physically identical [1, 2]. T-duality presents an important tool which shows the equivalence of different geometries and topologies. The useful T-duality procedure was first introduced by Buscher [3].

Mathematical realization of T-duality is given by Buscher T-dualization procedure [3], which is considered as standard one. There are also other frameworks in which we can

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[†]e-mail address: bnikolic@ipb.ac.rs

[‡]e-mail address: sazdovic@ipb.ac.rs

represent T-dualization which should agree with the Buscher procedure. It is double space formalism which was the subject of the articles about twenty years ago [4, 5, 6, 7, 8]. Double space is spanned by coordinates $Z^M = (x^\mu \ y_\mu)^T$ ($\mu = 0, 1, 2, \dots, D-1$), where x^μ and y_μ are the coordinates of the D -dimensional initial and T-dual space-time, respectively. Interest for this subject emerged recently with papers [9, 10, 11, 12, 13], where T-duality along some subset of d coordinates is considered as $O(d, d)$ symmetry transformation and [14, 15], where it is considered as permutation of d initial with corresponding d T-dual coordinates.

Until recently only T-duality along bosonic coordinates has been considered. Analyzing the gluon scattering amplitudes in $N = 4$ super Yang-Mills theory, a new kind of T-dual symmetry, fermionic T-duality, was discovered [16, 17]. It is a part of the dual superconformal symmetry which should be connected to integrability and it is valid just at string tree level. Mathematically, fermionic T-duality is realized within the same procedure as bosonic one, except that dualization is performed along fermionic variables. So, it can be considered as a generalization of Buscher T-duality. Fermionic T-duality consists in certain non-local redefinitions of the fermionic variables of the superstring mapping a supersymmetric background to another supersymmetric background. In Refs.[16, 17] it was shown that fermionic T-duality maps gluon scattering amplitudes in the original theory to an object very close to Wilson loops in the dual one. Calculation of gluon scattering amplitudes in the initial theory is equivalent to the calculation of Wilson loops in fermionic T-dual theory. Generalizing the idea of double space to the fermionic case we would get fermionic double space in which fermionic T-duality is a symmetry [18] which exchanges scattering amplitudes and Wilson loops. Fermionic double space can be also successfully applied in random lattice [19], where doubling of the supercoordinate was done. Relation between fermionic T-duality and open string noncommutativity was considered in Ref.[20].

Let us explain our motivation for fermionic T-duality. It is well known that T-duality is important feature in understanding the M-theory. In fact, five consistent superstring theories are connected by web of T and S dualities. We are going to pay attention to the T-duality, hoping that S-duality (which can be understood as transformation of dilaton background field also) can be later successfully incorporated into our procedure. If we start with arbitrary (of five consistent superstring) theory and find all corresponding T-dual theories we can achieve any of other four consistent superstring theories. But to obtain formulation of M-theory it is not enough. We must construct one theory which contain the initial theory and all corresponding T-dual ones.

In the bosonic case (which is substantially simpler than supersymmetric one) we have succeeded to realize such program. In Refs.[14, 15] we doubled all bosonic coordinates and showed that such theory contained the initial and all corresponding T-dual theories. We

can connect arbitrary two of these theories just replacing some initial coordinates x^a with corresponding T-dual ones y_a . This is equivalent with T-dualization along coordinates x^a . So, introducing double space T-duality ceases to be transformation which connects two physically equivalent theories but it becomes symmetry transformation in extended space with respect to permutation group. We proved this in the bosonic string case both for constant and for weakly curved background with linear dependence on coordinates.

Unfortunately, this is not enough for construction of M-theory, because the T-duality for superstrings is much more complicated then in the bosonic case. In Ref.[21] we have tried to extend such approach to the type II theories. In fact, doubling all bosonic coordinates we have unified types IIA, IIB as well as type II^* [22] (obtained by T-dualization along time-like direction) theories. There is an incompleteness in such approach. Doubling all bosonic coordinates, by simple permutations of initial with corresponding T-dual coordinates, we obtained all T-dual background fields except T-dual R-R field strength $F^{\alpha\beta}$. To obtain ${}_a F^{\alpha\beta}$ (the field strength after T-dualization along coordinates x^a) we need to introduce some additional assumptions. The explanation is that R-R field strength $F^{\alpha\beta}$ appears coupled with fermionic momenta π_α and $\bar{\pi}_\alpha$ along which we did not performed T-dualization and consequently we did not double these variables. It is an analogue of ij -term in approach of Refs.[9, 10] where x^i coordinates are not doubled.

Therefore, in the first step of our approach to the formulation of M-theory (unification of types II theories) we must include T-dualization along fermionic variables (π_α and $\bar{\pi}_\alpha$ in particular case). It means that we should doubled these fermionic variables, also. The present article represents a necessary step for understanding T-dualization along all fermionic coordinates in fermionic double space. We expect that final step in construction of M-theory will be unification of all theories obtained after T-dualization along all bosonic and all fermionic variables [18, 19]. In that case we should double all coordinates in superspace, anticipating that some super permutation will connect arbitrary two of our five consistent super symmetric string theories.

In this article we are going to double fermionic sector of type II theories adding to the coordinates θ^α and $\bar{\theta}^\alpha$ their fermionic T-duals, ϑ_α and $\bar{\vartheta}_\alpha$, where index α counts independent real components of the spinors, $\alpha = 1, 2, \dots, 16$. Rewriting T-dual transformation laws in terms of the double coordinates, $\Theta^A = (\theta^\alpha, \vartheta_\alpha)$ and $\bar{\Theta}^A = (\bar{\theta}^\alpha, \bar{\vartheta}_\alpha)$, we define the "fermionic generalized metric" \mathcal{F}_{AB} and the generalized currents $\bar{\mathcal{J}}_{+A}$ and \mathcal{J}_{-A} . The permutation matrix $\mathcal{T}^A{}_B$ exchanges $\bar{\theta}^\alpha$ and θ^α with their T-dual partners, $\bar{\vartheta}_\alpha$ and ϑ_α , respectively. From the requirement that fermionic T-dual coordinates, ${}^*\Theta^A = \mathcal{T}^A{}_B \Theta^B$ and ${}^*\bar{\Theta}^A = \mathcal{T}^A{}_B \bar{\Theta}^B$, have the same transformation law as initial ones, Θ^A and $\bar{\Theta}^A$, we obtain the expressions for fermionic T-dual generalized metric, ${}^*\mathcal{F}_{AB} = (\mathcal{T}\mathcal{F}\mathcal{T})_{AB}$, and T-dual currents, ${}^*\bar{\mathcal{J}}_{+A} = \mathcal{T}_A{}^B \bar{\mathcal{J}}_{+B}$ and ${}^*\mathcal{J}_{-A} = \mathcal{T}_A{}^B \mathcal{J}_{-B}$, in terms of the initial ones. These expressions produce the expression for fermionic T-dual NS-R fields and R-R field strength.

Expressions for fermionic T-dual metric and Kalb-Ramond field are obtained separately under some assumptions. We conclude that only symmetric part of R-R field strength, $F_s^{\alpha\beta} = \frac{1}{2}(F^{\alpha\beta} + F^{\beta\alpha})$, and symmetric part of its fermionic T-dual, ${}^sF_{\alpha\beta} = \frac{1}{2}({}^sF_{\alpha\beta} + {}^sF_{\beta\alpha})$, give contribution to the dilaton field transformation under fermionic T-duality. We also investigate the dilaton field in double space.

2 Type II superstring and fermionic T-duality

In this section we will introduce the action of type II superstring theory in pure spinor formulation and perform fermionic T-duality [16, 17, 20] using fermionic analogue of Buscher rules [3].

2.1 Action

In this manuscript we use the action of type II superstring theory in pure spinor formulation [23] up to the quadratic terms with constant background fields. Here we will derive the final form of the action which will be exploited in the further analysis. It corresponds to the actions used in Refs.[24, 25, 26, 27].

The sigma model action for type II superstring of Ref.[28] is of the form

$$S = S_0 + V_{SG}, \quad (2.1)$$

where S_0 is the action in the flat background

$$S_0 = \int_{\Sigma} d^2\xi \left(\frac{\kappa}{2} \eta^{mn} \eta_{\mu\nu} \partial_m x^\mu \partial_n x^\nu - \pi_\alpha \partial_- \theta^\alpha + \partial_+ \bar{\theta}^\alpha \bar{\pi}_\alpha \right) + S_\lambda + S_{\bar{\lambda}}, \quad (2.2)$$

and it is deformed by integrated form of the massless type II supergravity vertex operator

$$V_{SG} = \int_{\Sigma} d^2\xi (X^T)^M A_{MN} \bar{X}^N. \quad (2.3)$$

The vectors X^M and X^N are defined as

$$X^M = \begin{pmatrix} \partial_+ \theta^\alpha \\ \Pi_+^\mu \\ d_\alpha \\ \frac{1}{2} N^{\mu\nu} \end{pmatrix}, \quad \bar{X}^M = \begin{pmatrix} \partial_- \bar{\theta}^\alpha \\ \Pi_-^\mu \\ \bar{d}_\alpha \\ \frac{1}{2} \bar{N}^{\mu\nu} \end{pmatrix}, \quad (2.4)$$

and supermatrix A_{MN} is of the form

$$A_{MN} = \begin{pmatrix} A_{\alpha\beta} & A_{\alpha\nu} & E_\alpha^\beta & \Omega_{\alpha,\mu\nu} \\ A_{\mu\beta} & A_{\mu\nu} & \bar{E}_\mu^\beta & \Omega_{\mu,\nu\rho} \\ E_\nu^\alpha & E_\nu^\alpha & P^{\alpha\beta} & C^\alpha_{\mu\nu} \\ \Omega_{\mu\nu,\beta} & \Omega_{\mu\nu,\rho} & C_{\mu\nu}^\beta & S_{\mu\nu,\rho\sigma} \end{pmatrix}, \quad (2.5)$$

where notation and definitions are taken from Ref.[28]. The actions for pure spinors, S_λ and $S_{\bar{\lambda}}$, are free field actions and fully decoupled from the rest of action S_0 . The world sheet Σ is parameterized by $\xi^m = (\xi^0 = \tau, \xi^1 = \sigma)$ and $\partial_\pm = \partial_\tau \pm \partial_\sigma$. Bosonic part of superspace is spanned by coordinates x^μ ($\mu = 0, 1, 2, \dots, 9$), while the fermionic one is spanned by θ^α and $\bar{\theta}^\alpha$ ($\alpha = 1, 2, \dots, 16$). The variables π_α and $\bar{\pi}_\alpha$ are canonically conjugated momenta to θ^α and $\bar{\theta}^\alpha$, respectively. All spinors are Majorana-Weyl ones, which means that each of them has 16 independent real components. Matrix with superfields generally depends on x^μ , θ^α and $\bar{\theta}^\alpha$.

The superfields $A_{\mu\nu}$, $\bar{E}_\mu{}^\alpha$, $E^\alpha{}_\mu$ and $P^{\alpha\beta}$ are known as physical superfields, while the fields given in the first column and first row are auxiliary superfields because they can be expressed in terms of the physical ones. The rest ones, $\Omega_{\mu,\nu\rho}(\Omega_{\mu\nu,\rho})$, $C^\alpha{}_{\mu\nu}(C_{\mu\nu}{}^\alpha)$ and $S_{\mu\nu,\rho\sigma}$, are curvatures (field strengths) for physical superfields.

Demanding BRST invariance of vertex operator produces equations of motion

$$\Gamma_{\mu\nu\rho\sigma\lambda}^{\alpha\beta} D_\alpha A_{\beta\gamma} = 0, \quad \Gamma_{\mu\nu\rho\sigma\lambda}^{\alpha\beta} \bar{D}_\alpha A_{\gamma\beta} = 0. \quad (2.6)$$

In order to implement T-duality many restrictions should be imposed. For example, in bosonic case one should assume the existence of Killing vectors, which in fact means background fields independence of corresponding suitably selected coordinates. The idea is to avoid dependence on the coordinate x^μ and allow only dependence on the σ and τ derivatives of the coordinates, \dot{x}^μ and x'^μ . The case with explicit dependence on the coordinate require particular attention and has been considered in Ref.[29]. Similar simplifications must be imposed in consideration of the non-commutativity of the coordinates [30, 29].

A similar situation occurs in the supersymmetric case. We will additionally require that all physical superfields are constant (x independent) and restrict the analysis only to the quadratic terms in order to be able to perform T-dualization procedure. Because physical superfields are taken to be constant all three curvatures are equal to zero. In order to perform fermionic T-duality we must avoid explicit dependence on the fermionic coordinates θ^α and $\bar{\theta}^\alpha$ and allow only dependence on the σ and τ derivatives of these coordinates. In particular, auxiliary superfields can be taken to be zero (for expressions of auxiliary superfields see Ref.[28]).

Taking into account these assumptions there exists solution

$$\Pi_\pm^\mu \rightarrow \partial_\pm x^\mu, \quad d_\alpha \rightarrow \pi_\alpha, \quad \bar{d}_\alpha \rightarrow \bar{\pi}_\alpha, \quad (2.7)$$

and only nontrivial superfields take the form

$$A_{\mu\nu} = \kappa(\frac{1}{2}g_{\mu\nu} + B_{\mu\nu}), \quad E_\nu^\alpha = -\Psi_\nu^\alpha, \quad \bar{E}_\mu^\alpha = \bar{\Psi}_\mu^\alpha, \quad P^{\alpha\beta} = \frac{2}{\kappa}P^{\alpha\beta} = \frac{2}{\kappa}e^{\frac{\Phi}{2}}F^{\alpha\beta}, \quad (2.8)$$

where $g_{\mu\nu}$ is symmetric and $B_{\mu\nu}$ is antisymmetric tensor.

The final form of the vertex operator under these assumptions is

$$V_{SG} = \int_{\Sigma} d^2\xi \left[\kappa \left(\frac{1}{2} g_{\mu\nu} + B_{\mu\nu} \right) \partial_+ x^\mu \partial_- x^\nu - \pi_\alpha \Psi_\mu^\alpha \partial_- x^\mu + \partial_+ x^\mu \bar{\Psi}_\mu^\alpha \bar{\pi}_\alpha + \frac{2}{\kappa} \pi_\alpha P^{\alpha\beta} \pi_\beta \right]. \quad (2.9)$$

Consequently, the action S is of the form

$$S = \kappa \int_{\Sigma} d^2\xi \left[\partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu + \frac{1}{4\pi\kappa} \Phi R^{(2)} \right] + \int_{\Sigma} d^2\xi \left[-\pi_\alpha \partial_- (\theta^\alpha + \Psi_\mu^\alpha x^\mu) + \partial_+ (\bar{\theta}^\alpha + \bar{\Psi}_\mu^\alpha x^\mu) \bar{\pi}_\alpha + \frac{2}{\kappa} \pi_\alpha P^{\alpha\beta} \bar{\pi}_\beta \right], \quad (2.10)$$

where $G_{\mu\nu} = \eta_{\mu\nu} + g_{\mu\nu}$ and

$$\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}. \quad (2.11)$$

All terms containing pure spinors vanished because curvatures are zero under our assumption that physical superfields are constant. Actions S_λ and $S_{\bar{\lambda}}$ are fully decoupled from the rest action and can be neglected in the further analysis. The action, in its final form, is ghost independent.

Here we work both with type IIA and type IIB superstring theory. The difference is in the chirality of NS-R background fields and content of R-R sector. In NS-R sector there are two gravitino fields Ψ_μ^α and $\bar{\Psi}_\mu^\alpha$ which are Majorana-Weyl spinors of the opposite chirality in type IIA and same chirality in type IIB theory. The same feature stands for the pairs of spinors $(\theta^\alpha, \bar{\theta}^\alpha)$ and $(\pi_\alpha, \bar{\pi}_\alpha)$. The R-R field strength $F^{\alpha\beta}$ is expressed in terms of the antisymmetric tensors $F_{(k)} = F_{\mu_1\mu_2\ldots\mu_k}$ [1]

$$F^{\alpha\beta} = \sum_{k=0}^D \frac{1}{k!} F_{(k)} (C\Gamma_{(k)})^{\alpha\beta}, \quad \left[\Gamma_{(k)}^{\alpha\beta} = (\Gamma^{[\mu_1\ldots\mu_k]})^{\alpha\beta} \right] \quad (2.12)$$

where

$$\Gamma^{[\mu_1\mu_2\ldots\mu_k]} \equiv \Gamma^{\mu_1} \Gamma^{\mu_2} \ldots \Gamma^{\mu_k}, \quad (2.13)$$

is basis of completely antisymmetrized product of gamma matrices and C is charge conjugation operator. For more technical details regarding gamma matrices see the first reference in [1].

R-R field strength satisfies the chirality condition $\Gamma^{11}F = \pm F\Gamma^{11}$, where Γ^{11} is a product of gamma matrices in $D = 10$ dimensional space-time. The sign $+$ corresponds to type IIA while sign $-$ corresponds to type IIB superstring theory. Consequently, type IIA theory contains only even rank tensors $F_{(k)}$, while type IIB contains only odd rank tensors. For type IIA the independent tensors are $F_{(0)}$, $F_{(2)}$ and $F_{(4)}$, while independent tensors for type IIB are $F_{(1)}$, $F_{(3)}$ and self-dual part of $F_{(5)}$.

The fermionic part of the action (2.10) has the form of the first order theory. We want to eliminate the fermionic momenta and work with the action expressed in terms of

coordinates and their derivatives. So, on the equations of motion for fermionic momenta π_α and $\bar{\pi}_\alpha$,

$$\pi_\alpha = -\frac{\kappa}{2}\partial_+\left(\bar{\theta}^\beta + \bar{\Psi}_\mu^\beta x^\mu\right)(P^{-1})_{\beta\alpha}, \quad \bar{\pi}_\alpha = \frac{\kappa}{2}(P^{-1})_{\alpha\beta}\partial_-\left(\theta^\beta + \Psi_\mu^\beta x^\mu\right), \quad (2.14)$$

the action gets the form

$$\begin{aligned} S(\partial_\pm x, \partial_-\theta, \partial_+\bar{\theta}) &= \kappa \int_\Sigma d^2\xi \partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu + \frac{1}{4\pi} \int_\Sigma d^2\xi \Phi R^{(2)} \\ &+ \frac{\kappa}{2} \int_\Sigma d^2\xi \partial_+ \left(\bar{\theta}^\alpha + \bar{\Psi}_\mu^\alpha x^\mu\right) (P^{-1})_{\alpha\beta} \partial_- \left(\theta^\beta + \Psi_\nu^\beta x^\nu\right) \\ &= \kappa \int_\Sigma d^2\xi \partial_+ x^\mu \left[\Pi_{+\mu\nu} + \frac{1}{2} \bar{\Psi}_\mu^\alpha (P^{-1})_{\alpha\beta} \Psi_\nu^\beta \right] \partial_- x^\nu + \frac{1}{4\pi} \int_\Sigma d^2\xi \Phi R^{(2)} \\ &+ \frac{\kappa}{2} \int_\Sigma d^2\xi \left[\partial_+ \bar{\theta}^\alpha (P^{-1})_{\alpha\beta} \partial_- \theta^\beta + \partial_+ \bar{\theta}^\alpha (P^{-1} \Psi)_{\alpha\mu} \partial_- x^\mu + \partial_+ x^\mu (\bar{\Psi} P^{-1})_{\mu\alpha} \partial_- \theta^\alpha \right]. \end{aligned} \quad (2.15)$$

2.2 Fixing the chiral gauge invariance

In the above action θ^α appears only in the form $\partial_- \theta^\alpha$ and $\bar{\theta}^\alpha$ in the form $\partial_+ \bar{\theta}^\alpha$. This means that the theory has a local symmetry

$$\delta\theta^\alpha = \varepsilon^\alpha(\sigma^+), \quad \delta\bar{\theta}^\alpha = \bar{\varepsilon}^\alpha(\sigma^-), \quad (\sigma^\pm = \tau \pm \sigma). \quad (2.16)$$

We will treat this symmetry within BRST formalism. The corresponding BRST transformations are

$$s\theta^\alpha = c^\alpha(\sigma^+), \quad s\bar{\theta}^\alpha = \bar{c}^\alpha(\sigma^-), \quad (2.17)$$

where for each gauge parameter $\varepsilon^\alpha(\sigma^+)$ and $\bar{\varepsilon}^\alpha(\sigma^-)$ we introduced the ghost fields $c^\alpha(\sigma^+)$ and $\bar{c}^\alpha(\sigma^-)$, respectively. Here s is BRST nilpotent operator.

To fix gauge freedom we introduce gauge fermion with ghost number -1

$$\Psi = \frac{\kappa}{2} \int d^2\xi \left[\bar{C}_\alpha \left(\partial_+ \theta^\alpha + \frac{\alpha^{\alpha\beta}}{2} b_{+\beta} \right) + \left(\partial_- \bar{\theta}^\alpha + \frac{1}{2} \bar{b}_{-\beta} \alpha^{\beta\alpha} \right) C_\alpha \right], \quad (2.18)$$

where $\alpha^{\alpha\beta}$ is arbitrary non singular matrix, \bar{C}_α and C_α are antighost fields, while $b_{+\alpha}$ and $\bar{b}_{-\alpha}$ are Nakanishi-Lautrup auxillary fields which satisfy

$$sC_\alpha = b_{+\alpha}, \quad s\bar{C}_\alpha = \bar{b}_{-\alpha}, \quad sb_{+\alpha} = 0 \quad s\bar{b}_{-\alpha} = 0. \quad (2.19)$$

BRST transformation of gauge fermion Ψ produces the gauge fixed and Fadeev-Popov action

$$\begin{aligned} s\Psi &= S_{gf} + S_{FP}, \\ S_{gf} &= \frac{\kappa}{2} \int d^2\xi \left[\bar{b}_{-\alpha} \partial_+ \theta^\alpha + \partial_- \bar{\theta}^\alpha b_{+\alpha} + \bar{b}_{-\alpha} \alpha^{\alpha\beta} b_{+\beta} \right], \\ S_{FD} &= \frac{\kappa}{2} \int d^2\xi \left[\bar{C}_\alpha \partial_+ c^\alpha + (\partial_- \bar{c}^\alpha) C_\alpha \right]. \end{aligned} \quad (2.20)$$

The Fadeev-Popov action is decoupled from the rest and, consequently, it can be omitted in further analysis. On the equations of motion for b -fields

$$b_{+\alpha} = -(\alpha^{-1})_{\alpha\beta} \partial_+ \theta^\alpha, \quad \bar{b}_{-\alpha} = -\partial_- \bar{\theta}^\beta (\alpha^{-1})_{\beta\alpha}, \quad (2.21)$$

we obtain the final form of the BRST gauge fixed action

$$S_{gf} = -\frac{\kappa}{2} \int d^2 \xi \partial_- \bar{\theta}^\alpha (\alpha^{-1})_{\alpha\beta} \partial_+ \theta^\beta. \quad (2.22)$$

2.3 Fermionic T-duality

We will perform fermionic T-duality using fermionic version of Buscher procedure similarly to Refs.[20] where we worked without chiral gauge fixing. After introducing S_{gf} the action still has a global shift symmetry in θ^α and $\bar{\theta}^\alpha$ directions. We introduce gauge fields v_\pm^α and \bar{v}_\pm^α and replace ordinary world-sheet derivatives with covariant ones

$$\partial_\pm \theta^\alpha \rightarrow D_\pm \theta^\alpha \equiv \partial_\pm \theta^\alpha + v_\pm^\alpha, \quad \partial_\pm \bar{\theta}^\alpha \rightarrow D_\pm \bar{\theta}^\alpha \equiv \partial_\pm \bar{\theta}^\alpha + \bar{v}_\pm^\alpha. \quad (2.23)$$

In order to make the fields v_\pm^α and \bar{v}_\pm^α to be unphysical we add the following terms in the action

$$S_{gauge}(\vartheta, v_\pm, \bar{\vartheta}, \bar{v}_\pm) = \frac{1}{2} \kappa \int_\Sigma d^2 \xi \bar{\vartheta}_\alpha (\partial_+ v_-^\alpha - \partial_- v_+^\alpha) + \frac{1}{2} \kappa \int_\Sigma d^2 \xi (\partial_+ \bar{v}_-^\alpha - \partial_- \bar{v}_+^\alpha) \vartheta_\alpha, \quad (2.24)$$

where ϑ_α and $\bar{\vartheta}_\alpha$ are Lagrange multipliers. The full gauge invariant action is of the form

$$S_{inv}(x, \theta, \bar{\theta}, \vartheta, \bar{\vartheta}, v_\pm, \bar{v}_\pm) = S(\partial_\pm x, D_- \theta, D_+ \bar{\theta}) + S_{gf}(D_- \theta, D_+ \bar{\theta}) + S_{gauge}(\vartheta, \bar{\vartheta}, v_\pm, \bar{v}_\pm). \quad (2.25)$$

Fixing θ^α and $\bar{\theta}^\alpha$ to zero we obtain the gauge fixed action

$$\begin{aligned} S_{fix} = & \kappa \int_\Sigma d^2 \xi \partial_+ x^\mu \left[\Pi_{+\mu\nu} + \frac{1}{2} \bar{\Psi}_\mu^\alpha (P^{-1})_{\alpha\beta} \Psi_\nu^\beta \right] \partial_- x^\nu + \frac{1}{4\pi} \int_\Sigma d^2 \xi \Phi R^{(2)} \\ & + \frac{\kappa}{2} \int_\Sigma \left[\bar{v}_+^\alpha (P^{-1})_{\alpha\beta} v_-^\beta + \bar{v}_+^\alpha (P^{-1})_{\alpha\beta} \Psi_\nu^\beta \partial_- x^\nu + \partial_+ x^\mu \bar{\Psi}_\mu^\alpha (P^{-1})_{\alpha\beta} v_-^\beta - \bar{v}_-^\alpha (\alpha^{-1})_{\alpha\beta} v_+^\beta \right] \\ & + \frac{\kappa}{2} \int_\Sigma d^2 \xi \bar{\vartheta}_\alpha (\partial_+ v_-^\alpha - \partial_- v_+^\alpha) + \frac{\kappa}{2} \int_\Sigma d^2 \xi (\partial_+ \bar{v}_-^\alpha - \partial_- \bar{v}_+^\alpha) \vartheta_\alpha. \end{aligned} \quad (2.26)$$

Varying the above action with respect to the Lagrange multipliers ϑ_α and $\bar{\vartheta}_\alpha$ we obtain the initial action (2.15) because

$$\partial_+ v_-^\alpha - \partial_- v_+^\alpha = 0 \implies v_\pm^\alpha = \partial_\pm \theta^\alpha, \quad \partial_+ \bar{v}_-^\alpha - \partial_- \bar{v}_+^\alpha = 0 \implies \bar{v}_\pm^\alpha = \partial_\pm \bar{\theta}^\alpha. \quad (2.27)$$

The equations of motion for v_\pm^α and \bar{v}_\pm^α give

$$\bar{v}_-^\alpha = \partial_- \bar{\vartheta}_\beta \alpha^{\beta\alpha}, \quad \bar{v}_+^\alpha = \partial_+ \bar{\vartheta}_\beta P^{\beta\alpha} - \partial_+ x^\mu \bar{\Psi}_\mu^\alpha, \quad (2.28)$$

$$v_+^\alpha = -\alpha^{\alpha\beta} \partial_+ \vartheta_\beta, \quad v_-^\alpha = -P^{\alpha\beta} \partial_- \vartheta_\beta - \Psi_\mu^\alpha \partial_- x^\mu. \quad (2.29)$$

Substituting these expressions in the action S_{fix} we obtain the fermionic T-dual action

$$\begin{aligned} {}^*S(\partial_\pm x, \partial_- \vartheta, \partial_+ \bar{\vartheta}) &= \kappa \int_\Sigma d^2 \xi \partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu + \frac{1}{4\pi} \int_\Sigma d^2 \xi {}^* \Phi R^{(2)}, \\ &+ \frac{\kappa}{2} \int_\Sigma d^2 \xi \left[\partial_+ \bar{\vartheta}_\alpha P^{\alpha\beta} \partial_- \vartheta_\beta - \partial_+ x^\mu \bar{\Psi}_\mu^\alpha \partial_- \vartheta_\alpha + \partial_+ \bar{\vartheta}_\alpha \Psi_\mu^\alpha \partial_- x^\mu - \partial_- \bar{\vartheta}_\alpha \alpha^{\alpha\beta} \partial_+ \vartheta_\beta \right]. \end{aligned} \quad (2.30)$$

It should be in the form of the initial action (2.15)

$$\begin{aligned} {}^*S &= \kappa \int_\Sigma d^2 \xi \partial_+ x^\mu \left[{}^* \Pi_{+\mu\nu} + \frac{1}{2} {}^* \Psi^{\alpha\mu} ({}^* P^{-1})^{\alpha\beta} {}^* \Psi_{\beta\nu} \right] \partial_- x^\nu + \frac{1}{4\pi} \int_\Sigma d^2 \xi {}^* \Phi R^{(2)} \\ &+ \frac{\kappa}{2} \int_\Sigma d^2 \xi \left[\partial_+ \bar{\vartheta}_\alpha ({}^* P^{-1})^{\alpha\beta} \partial_- \vartheta_\beta + \partial_+ x^\mu ({}^* \bar{\Psi} {}^* P^{-1})_\mu^\alpha \partial_- \vartheta_\alpha + \partial_+ \bar{\vartheta}_\alpha ({}^* P^{-1} {}^* \Psi)_\mu^\alpha \partial_- x^\mu \right] \\ &- \frac{\kappa}{2} \int_\Sigma d^2 \xi \partial_- \bar{\vartheta}_\alpha ({}^* \alpha^{-1})^{\alpha\beta} \partial_+ \vartheta_\beta, \end{aligned} \quad (2.31)$$

and so we get

$${}^* \Psi_{\alpha\mu} = (P^{-1} \Psi)_{\alpha\mu}, \quad {}^* \bar{\Psi}_{\mu\alpha} = -(\bar{\Psi} P^{-1})_{\mu\alpha}, \quad (2.33)$$

$${}^* P_{\alpha\beta} = (P^{-1})_{\alpha\beta}, \quad {}^* \alpha_{\alpha\beta} = (\alpha^{-1})_{\alpha\beta}. \quad (2.34)$$

From the condition

$${}^* \Pi_{+\mu\nu} + \frac{1}{2} {}^* \bar{\Psi}_{\alpha\mu} ({}^* P^{-1})^{\alpha\beta} {}^* \Psi_{\beta\nu} = \Pi_{+\mu\nu}, \quad (2.35)$$

we read the fermionic T-dual metric and Kalb-Ramond field

$$\begin{aligned} {}^* G_{\mu\nu} &= G_{\mu\nu} + \frac{1}{2} [(\bar{\Psi} P^{-1} \Psi)_{\mu\nu} + (\bar{\Psi} P^{-1} \Psi)_{\nu\mu}], \\ {}^* B_{\mu\nu} &= B_{\mu\nu} + \frac{1}{4} [(\bar{\Psi} P^{-1} \Psi)_{\mu\nu} - (\bar{\Psi} P^{-1} \Psi)_{\nu\mu}]. \end{aligned} \quad (2.36)$$

Dilaton transformation under fermionic T-duality will be presented in the section 4. Let us note that two successive dualizations give the initial background fields.

The T-dual transformation laws are connection between initial and T-dual coordinates. We can obtain them combining the different solutions of equations of motion for v_\pm^α and \bar{v}_\pm^α (2.27) and (2.28)-(2.29)

$$\partial_- \theta^\alpha \cong -P^{\alpha\beta} \partial_- \vartheta_\beta - \Psi_\mu^\alpha \partial_- x^\mu, \quad \partial_+ \bar{\theta}^\alpha \cong \partial_+ \bar{\vartheta}_\beta P^{\beta\alpha} - \partial_+ x^\mu \bar{\Psi}_\mu^\alpha, \quad (2.37)$$

$$\partial_+ \theta^\alpha \cong -\alpha^{\alpha\beta} \partial_+ \vartheta_\beta, \quad \partial_- \bar{\theta}^\alpha \cong \partial_- \bar{\vartheta}_\beta \alpha^{\beta\alpha}. \quad (2.38)$$

Here the symbol \cong denotes the T-duality relation. From these relations we can obtain inverse transformation rules

$$\partial_- \vartheta_\alpha \cong -(P^{-1})_{\alpha\beta} \partial_- \theta^\beta - (P^{-1})_{\alpha\beta} \Psi_\mu^\beta \partial_- x^\mu, \quad \partial_+ \bar{\vartheta}_\alpha \cong \partial_+ \bar{\theta}^\beta (P^{-1})_{\beta\alpha} + \partial_+ x^\mu \bar{\Psi}_\mu^\beta (P^{-1})_{\beta\alpha}, \quad (2.39)$$

$$\partial_+ \vartheta_\alpha \cong -(\alpha^{-1})_{\alpha\beta} \partial_+ \theta^\beta, \quad \partial_- \bar{\vartheta}_\alpha \cong \partial_- \bar{\theta}^\beta (\alpha^{-1})_{\beta\alpha}. \quad (2.40)$$

Note that without gauge fixing in subsection 2.2, instead expressions for \bar{v}_-^α and v_+^α (first relations of (2.28) and (2.29)), we would have only constraints on the T-dual variables, $\partial_- \bar{\vartheta}_\alpha = 0$ and $\partial_+ \vartheta_\alpha = 0$. Consequently, integration over v_\pm^α and \bar{v}_\pm^α would be singular and we would lose the part of T-dual transformations (2.38) and (2.40).

3 Fermionic T-dualization in fermionic double space

In this section we will extend the meaning of the double space. We will introduce double fermionic space adding to the fermionic coordinates, θ^α and $\bar{\theta}^\alpha$, the fermionic T-dual ones, ϑ_α and $\bar{\vartheta}_\alpha$. Then we will show that fermionic T-dualization can be represented as permutation of the appropriate fermionic coordinates and their T-dual partners.

3.1 Transformation laws in fermionic double space

In the same way as the double bosonic coordinates were introduced [4, 14, 15], we double both fermionic coordinate as

$$\Theta^A = \begin{pmatrix} \theta^\alpha \\ \vartheta_\alpha \end{pmatrix}, \quad \bar{\Theta}^A = \begin{pmatrix} \bar{\theta}^\alpha \\ \bar{\vartheta}_\alpha \end{pmatrix}. \quad (3.1)$$

Each double coordinate has 32 real components. In terms of the double fermionic coordinates the transformation laws, (2.37)-(2.40), can be rewritten in the form

$$\partial_- \Theta^A \cong -\Omega^{AB} [\mathcal{F}_{BC} \partial_- \Theta^C + \mathcal{J}_{-B}], \quad \partial_+ \bar{\Theta}^A \cong [\partial_+ \bar{\Theta}^C \mathcal{F}_{CB} + \bar{\mathcal{J}}_{+B}] \Omega^{BA}, \quad (3.2)$$

$$\partial_+ \Theta^A \cong -\Omega^{AB} \mathcal{A}_{BC} \partial_+ \Theta^C, \quad \partial_- \bar{\Theta}^A \cong \partial_- \bar{\Theta}^C \mathcal{A}_{CB} \Omega^{BA}, \quad (3.3)$$

where "fermionic generalized metric" \mathcal{F}_{AB} has the form

$$\mathcal{F}_{AB} = \begin{pmatrix} (P^{-1})_{\alpha\beta} & 0 \\ 0 & P^{\gamma\delta} \end{pmatrix}, \quad (3.4)$$

and

$$\mathcal{A}_{AB} = \begin{pmatrix} (\alpha^{-1})_{\alpha\beta} & 0 \\ 0 & \alpha^{\gamma\delta} \end{pmatrix}. \quad (3.5)$$

\mathcal{F}_{AB} is bosonic variable but we put the name fermionic because it appears in the case of fermionic T-duality.

The double currents, $\bar{\mathcal{J}}_{+A}$ and \mathcal{J}_{-A} , are fermionic variables of the form

$$\bar{\mathcal{J}}_{+A} = \begin{pmatrix} (\bar{\Psi} P^{-1})_{\mu\alpha} \partial_+ x^\mu \\ -\bar{\Psi}_\mu^\alpha \partial_+ x^\mu \end{pmatrix}, \quad \mathcal{J}_{-A} = \begin{pmatrix} (P^{-1} \Psi)_{\alpha\mu} \partial_- x^\mu \\ \Psi_\mu^\alpha \partial_- x^\mu \end{pmatrix}, \quad (3.6)$$

while the matrix Ω^{AB} is constant

$$\Omega^{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (3.7)$$

where identity matrix is 16×16 . By straightforward calculation we can prove the relations

$$\Omega^2 = 1, \quad \det \mathcal{F}_{AB} = 1. \quad (3.8)$$

Consistency of the transformation laws (3.2) produces

$$(\Omega \mathcal{F})^2 = 1, \quad \mathcal{J}_- = \mathcal{F} \Omega \mathcal{J}_-, \quad \bar{\mathcal{J}}_+ = -\bar{\mathcal{J}}_+ \Omega \mathcal{F}. \quad (3.9)$$

3.2 Double action

It is well known that equations of motion of initial theory are Bianchi identities in T-dual picture and vice versa. As a consequence of the identities

$$\partial_+ \partial_- \Theta^A - \partial_- \partial_+ \Theta^A = 0, \quad \partial_+ \partial_- \bar{\Theta}^A - \partial_- \partial_+ \bar{\Theta}^A = 0, \quad (3.10)$$

known as Bianchi identities, and relations (3.2) and (3.3), we obtain the consistency conditions

$$\partial_+ (\mathcal{F}_{AB} \partial_- \Theta^B + J_{-A}) - \partial_- (\mathcal{A}_{AB} \partial_+ \Theta^B) = 0, \quad (3.11)$$

$$\partial_- (\partial_+ \bar{\Theta}^B \mathcal{F}_{BA} + \bar{J}_{+A}) - \partial_+ (\partial_- \bar{\Theta}^B \mathcal{A}_{BA}) = 0. \quad (3.12)$$

The equations (3.11) and (3.12) are equations of motion of the following action

$$\begin{aligned} S_{double}(\Theta, \bar{\Theta}) &= \\ &= \frac{\kappa}{2} \int d^2 \xi [\partial_+ \bar{\Theta}^A \mathcal{F}_{AB} \partial_- \Theta^B + \bar{J}_{+A} \partial_- \Theta^A + \partial_+ \bar{\Theta}^A \mathcal{J}_{-A} - \partial_- \bar{\Theta}^A \mathcal{A}_{AB} \partial_+ \Theta^B + L(x)], \end{aligned} \quad (3.13)$$

where $L(x)$ is arbitrary functional of the bosonic coordinates.

3.3 Fermionic T-dualization of type II superstring theory as permutation of fermionic coordinates in double space

In order to exchange θ^α with ϑ_α and $\bar{\theta}$ with $\bar{\vartheta}_\alpha$, let us introduce the permutation matrix

$$\mathcal{T}^A{}_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (3.14)$$

so that double T-dual coordinates are

$$*\Theta^A = \mathcal{T}^A{}_B \Theta^B, \quad *\bar{\Theta}^A = \mathcal{T}^A{}_B \bar{\Theta}^B. \quad (3.15)$$

We demand that T-dual transformation laws for double T-dual coordinates ${}^*\Theta^A$ and ${}^*\bar{\Theta}^A$ have the same form as for initial ones Θ^A and $\bar{\Theta}^A$ (3.2) and (3.3)

$$\partial_- {}^*\Theta^A \cong -\Omega^{AB} [{}^*\mathcal{F}_{BC} \partial_- {}^*\Theta^C + {}^*\mathcal{J}_{-B}] , \quad \partial_+ {}^*\bar{\Theta}^A \cong [\partial_+ {}^*\bar{\Theta}^C {}^*\mathcal{F}_{CB} + {}^*\bar{\mathcal{J}}_{+B}] \Omega^{BA} , \quad (3.16)$$

$$\partial_+ {}^*\Theta^A \cong -\Omega^{AB} {}^*\mathcal{A}_{BC} \partial_+ {}^*\Theta^C , \quad \partial_- {}^*\bar{\Theta}^A \cong \partial_- {}^*\bar{\Theta}^C {}^*\mathcal{A}_{CB} \Omega^{BA} . \quad (3.17)$$

Then the fermionic T-dual "generalized metric" ${}^*\mathcal{F}_{AB}$ and T-dual currents, ${}^*\bar{\mathcal{J}}_{+A}$ and ${}^*\mathcal{J}_{-A}$, with the help of (3.15) and (3.2), can be expressed in terms of initial ones

$${}^*\mathcal{F}_{AB} = \mathcal{T}_A^C \mathcal{F}_{CD} \mathcal{T}_B^D , \quad {}^*\bar{\mathcal{J}}_{+A} = \mathcal{T}_A^B \bar{\mathcal{J}}_{+B} , \quad {}^*\mathcal{J}_{-A} = \mathcal{T}_A^B \mathcal{J}_{-B} . \quad (3.18)$$

The matrix \mathcal{A}_{AB} transforms as

$${}^*\mathcal{A}_{AB} = \mathcal{T}_A^C \mathcal{A}_{CD} \mathcal{T}_B^D = (\mathcal{A}^{-1})_{AB} . \quad (3.19)$$

Note that, as well as bosonic case, double space action (3.13) has global symmetry under transformations (3.15) if the conditions (3.18) are satisfied.

From the first relation in (3.18) we obtain the form of the fermionic T-dual R-R background field

$${}^*P_{\alpha\beta} = (P^{-1})_{\alpha\beta} , \quad (3.20)$$

while from the second and third equation we obtain the form of the fermionic T-dual NS-R background fields

$${}^*\Psi_{\alpha\mu} = (P^{-1})_{\alpha\beta} \Psi_\mu^\beta , \quad {}^*\bar{\Psi}_{\alpha\mu} = -\bar{\Psi}_\mu^\beta (P^{-1})_{\beta\alpha} . \quad (3.21)$$

The non singular matrix $\alpha^{\alpha\beta}$ transforms as

$$({}^*\alpha)_{\alpha\beta} = (\alpha^{-1})_{\alpha\beta} . \quad (3.22)$$

The expressions (3.20)-(3.22) are in full agreement with the relations (2.33) and (2.34) obtained by the standard fermionic Buscher procedure. Consequently, we showed that permutation of fermionic coordinates defined in (3.14) and (3.15) completely reproduces fermionic T-dual R-R and NS-R background fields.

3.4 Fermionic T-dual metric ${}^*G_{\mu\nu}$ and Kalb-Ramond field ${}^*B_{\mu\nu}$

The expression $\Pi_{+\mu\nu} + \frac{1}{2} \Psi_\mu^\alpha (P^{-1})_{\alpha\beta} \Psi_\nu^\beta$ appears in the action (2.15) coupled with $\partial_\pm x^\mu$, along which we do not T-dualize. It is an analogue of ij -term of Refs.[9, 10] where x^i coordinates are not T-dualized, and $\alpha\beta$ -term in [21] where fermionic directions are undualized.

Taking into account the form of the doubled action (3.13) we suppose that term $L(x)$ has the form

$$L(x) = 2\partial_+ x^\mu (\Pi_{+\mu\nu} + {}^*\Pi_{+\mu\nu}) \partial_- x^\nu \equiv \mathcal{L} + {}^*\mathcal{L}, \quad (3.23)$$

where $\Pi_{+\mu\nu}$ is defined in (2.11) and ${}^*\Pi_{+\mu\nu}$ is fermionic T-dual which we are going to find. This term should be invariant under T-dual transformation

$${}^*\mathcal{L} = \mathcal{L} + \Delta\mathcal{L}. \quad (3.24)$$

Using the fact that two successive T-dualization are identity transformation, we obtain

$$\mathcal{L} = {}^*\mathcal{L} + {}^*\Delta\mathcal{L}. \quad (3.25)$$

Combining last two relations we get

$${}^*\Delta\mathcal{L} = -\Delta\mathcal{L}. \quad (3.26)$$

If $\Delta\mathcal{L} = 2\partial_+ x^\mu \Delta_{\mu\nu} \partial_- x^\nu$, we obtain the condition for $\Delta_{\mu\nu}$

$${}^*\Delta_{\mu\nu} = -\Delta_{\mu\nu}. \quad (3.27)$$

Using the relations (2.33) and (2.34) we realize that, up to multiplication constant, combination

$$\Delta_{\mu\nu} = \bar{\Psi}_\mu^\alpha (P^{-1})_{\alpha\beta} \Psi_\nu^\beta, \quad (3.28)$$

satisfies the condition (3.27). So, we conclude that

$${}^*\Pi_{+\mu\nu} = \Pi_{+\mu\nu} + c\bar{\Psi}_\mu^\alpha (P^{-1})_{\alpha\beta} \Psi_\nu^\beta, \quad (3.29)$$

where c is an arbitrary constant. For $c = \frac{1}{2}$ we obtain the relations (2.36). So, in double space formulation the fermionic T-dual NS-NS background fields can be obtained up to an arbitrary constant under assumption that two successive T-dualizations produce initial action.

4 Dilaton field in double fermionic space

Dilaton field transformation under fermionic T-duality is considered [16]. Here we will discuss some new features of dilaton transformation under fermionic T-duality as well as the dilaton field in fermionic double space.

Because the dilaton transformation has quantum origin we start with the path integral for the gauge fixed action given in Eq.(2.26)

$$Z = \int d\bar{v}_+^\alpha d\bar{v}_-^\alpha dv_+^\alpha dv_-^\alpha d\bar{\vartheta}_\alpha d\vartheta_\alpha e^{i S_{fix}(v_\pm, \bar{v}_\pm, \partial_\pm \vartheta, \partial_\pm \bar{\vartheta})}. \quad (4.1)$$

For constant background case, after integration over the fermionic gauge fields \bar{v}_\pm^α and v_\pm^α , we obtain the generating functional Z in the form

$$Z = \int d\bar{\vartheta}_\alpha d\vartheta_\alpha \det [(P^{-1}\alpha^{-1})_{\alpha\beta}] e^{i \star S(\vartheta, \bar{\vartheta})}, \quad (4.2)$$

where $\star S(\vartheta, \bar{\vartheta})$ is T-dual action given in Eq.(2.30). We are able to perform such integration thank to the facts that we fixed the gauge in subsection 2.2.

Note that here we multiplied with determinants of P^{-1} and α^{-1} because we integrate over Grassman fields v_\pm^α and \bar{v}_\pm^α . We can choose that $\det \alpha = 1$, and the generating functional gets the form

$$Z = \int d\bar{\vartheta}_\alpha d\vartheta_\alpha \det [(P^{-1})_{\alpha\beta}] e^{i \star S(\vartheta, \bar{\vartheta})}. \quad (4.3)$$

This produces the fermionic T-dual transformation of dilaton field

$$\star \Phi = \Phi + \ln \det [(P^{-1})_{\alpha\beta}] = \Phi - \ln \det P^{\alpha\beta}. \quad (4.4)$$

Let us calculate $\det P^{\alpha\beta}$ using the expression

$$(PP_s^{-1}P^T)^{\alpha\beta} = P_s^{\alpha\beta} - P_a^{\alpha\gamma}(P_s^{-1})_{\gamma\delta}P_a^{\delta\beta}, \quad (4.5)$$

where we introduce the symmetric and antisymmetric parts for initial background fields

$$P_s^{\alpha\beta} = \frac{1}{2} (P^{\alpha\beta} + P^{\beta\alpha}), \quad P_a^{\alpha\beta} = \frac{1}{2} (P^{\alpha\beta} - P^{\beta\alpha}), \quad (4.6)$$

and similar expressions for T-dual background fields, $\star P_{\alpha\beta}^s$ and $\star P_{\alpha\beta}^a$. Taking into account that

$$(P \cdot \star P)^\alpha{}_\beta = \delta^\alpha{}_\beta, \quad (4.7)$$

we obtain

$$P_s^{\alpha\gamma} \star P_{\gamma\beta}^s + P_a^{\alpha\gamma} \star P_{\gamma\beta}^a = \delta^\alpha{}_\beta, \quad P_s^{\alpha\gamma} \star P_{\gamma\beta}^a + P_a^{\alpha\gamma} \star P_{\gamma\beta}^s = 0. \quad (4.8)$$

From these two equations we obtain

$$\star P_{\alpha\beta}^s = [(P_s - P_a P_s^{-1} P_a)^{-1}]_{\alpha\beta}, \quad (4.9)$$

and, consequently, we have

$$(PP_s^{-1}P^T)^{\alpha\beta} = [(\star P_s)^{-1}]^{\alpha\beta}. \quad (4.10)$$

Taking determinant of the left and right-hand side of the above equation we get

$$\det P^{\alpha\beta} = \sqrt{\frac{\det P_s}{\det \star P_s}}, \quad (4.11)$$

which produces

$$*\Phi = \Phi - \ln \sqrt{\frac{\det P_s}{\det *P_s}}. \quad (4.12)$$

Using the fact that $P^{\alpha\beta} = e^{\frac{\Phi}{2}} F^{\alpha\beta}$ and $*P^{\alpha\beta} = e^{\frac{* \Phi}{2}} *F^{\alpha\beta}$, fermionic T-dual transformation law for dilaton takes the form

$$*\Phi = \Phi - \ln \sqrt{e^{8(\Phi - *\Phi)} \frac{\det F_s}{\det *F_s}}, \quad (4.13)$$

and finally we have

$$*\Phi = \Phi + \frac{1}{6} \ln \frac{\det F_s}{\det *F_s}. \quad (4.14)$$

It is obvious that two successive T-dualizations act as identity transformation

$$**\Phi = \Phi. \quad (4.15)$$

We can conclude that only symmetric parts of the R-R field strengths give contribution to the transformation of dilaton field under fermionic T-duality. In type IIA superstring theory R-R field strength $F^{\alpha\beta}$ contains tensors F_0^A , $F_{\mu\nu}^A$ and $F_{\mu\nu\rho\lambda}^A$, while in type IIB $F^{\alpha\beta}$ contains F_μ^B , $F_{\mu\nu\rho}^B$ and self dual part of $F_{\mu\nu\rho\lambda\omega}^B$. Using the conventions for gamma matrices from the appendix of the first reference in [1] (see Appendix A), we conclude that symmetric part of $F^{\alpha\beta}$ in type IIA contains scalar F_0^A and 2-rank tensor $F_{\mu\nu}^A$, while in type IIB superstring theory it contains 1-rank F_μ^B and self dual part of 5-rank tensor $F_{\mu\nu\rho\lambda\omega}^B$.

Let us write the path integral for double action (3.13)

$$Z_{double} = \int d\Theta^A d\bar{\Theta}^A e^{iS_{double}(\Theta, \bar{\Theta})}. \quad (4.16)$$

Because $\det \mathcal{F} = 1$ and $\det \mathcal{A} = 1$ we obtain that dilaton field in double space is invariant under fermionic T-duality. Consequently, a new dilaton should be introduced (see [14, 15]), invariant under T-duality transformations. Because of the relation (4.15) we define the T-duality invariant dilaton as

$$\Phi_{inv} = \frac{1}{2} (*\Phi + \Phi) = \Phi + \frac{1}{12} \ln \frac{\det F_s}{\det *F_s}, \quad *\Phi_{inv} = \Phi_{inv}. \quad (4.17)$$

5 Concluding remarks

In this article we considered the fermionic T-duality of the type II superstring theory using the double space approach. We used the action of the type II superstring theory in pure spinor formulation neglecting ghost terms and keeping all terms up to the quadratic ones which means that all background fields are constant.

Using equations of motion with respect to the fermionic momenta π_α and $\bar{\pi}_\alpha$ we eliminate them from the action. We obtained the action expressed in terms of the derivatives $\partial_\pm x^\mu$, $\partial_- \theta^\alpha$ and $\partial_+ \bar{\theta}^\alpha$, where θ^α and $\bar{\theta}^\alpha$ are fermionic coordinates. Because θ^α appears in the action in the form $\partial_- \theta^\alpha$ and $\bar{\theta}^\alpha$ in the form $\partial_+ \bar{\theta}^\alpha$, there is a local chiral gauge symmetry with parameters depending on $\sigma^\pm = \tau \pm \sigma$. We fixed this gauge invariance using BRST approach.

Using the Buscher approach we performed fermionic T-duality procedure and obtained the form of the fermionic T-dual background fields. It is obvious that two successive fermionic T-dualization produces initial theory i.e. they are equivalent to the identity transformation.

In the central point of the article we generalize the idea of double space and show that fermionic T-duality can be represented as permutation in fermionic double space. In the bosonic case double space spanned by coordinates $Z^M = (x^\mu, y_\mu)$ can be obtained adding T-dual coordinates y_μ to the initial ones x^μ . Using analogy with the bosonic case we introduced double fermionic space doubling the initial coordinates θ^α and $\bar{\theta}^\alpha$ with their fermionic T-duals, ϑ_α and $\bar{\vartheta}_\alpha$. Double fermionic space is spanned by the coordinates $\Theta^A = (\theta^\alpha, \vartheta_\alpha)$ and $\bar{\Theta}^A = (\bar{\theta}^\alpha, \bar{\vartheta}_\alpha)$.

T-dual transformation laws and their inverse ones are rewritten in fermionic double space by single relation introducing the fermionic generalized metric \mathcal{F}_{AB} and currents \mathcal{J}_{-A} and $\bar{\mathcal{J}}_{+A}$. Demanding that transformation laws for fermionic T-dual double coordinates, ${}^*\Theta^A = \mathcal{T}^A_B \Theta^B$ and ${}^*\bar{\Theta}^A = \mathcal{T}^A_B \bar{\Theta}^B$, are of the same form as those for Θ^A and $\bar{\Theta}^A$, we obtained fermionic T-dual generalized metric ${}^*\mathcal{F}_{AB}$ and currents ${}^*\mathcal{J}_{-A}$ and ${}^*\bar{\mathcal{J}}_{+A}$. These transformations act as symmetry transformations of the double action (3.13). They produce the form of the fermionic T-dual NS-R and R-R background fields which are in full accordance with the results obtained by standard Buscher procedure.

The expressions for T-dual metric ${}^*G_{\mu\nu}$ and Kalb-Ramond field ${}^*B_{\mu\nu}$ cannot be found from double space formalism because they do not appear in the T-dual transformation laws. These expressions, up to arbitrary constant, are obtained assuming that two successive T-dualization act as identity transformation.

We considered transformation of dilaton field under fermionic T-duality. We derived the transformation law for dilaton field and concluded that just symmetric parts of R-R field strengths, $F_s^{\alpha\beta}$ and ${}^*F_{\alpha\beta}^s$, affected the dilaton transformation law. This means that in the case of type IIA scalar and 2-rank tensor have influence on the dilaton transformation, while in the case of type IIB 1-rank tensor and self-dual part of 5-rank tensor take that role.

Therefore, we extended T-dualization in double space to the fermionic case. We proved that permutation of fermionic coordinates with corresponding T-dual ones in double space is equivalent to the fermionic T-duality along initial coordinates θ^α and $\bar{\theta}^\alpha$.

A Gamma matrices

In the appendix of the first reference of [1] one specific representation of gamma matrices is given. Here we will calculate the transpositions of basis matrices $(C\Gamma_{(k)})^{\alpha\beta}$ for $k = 1, 2, 3, 4, 5$, where C is charge conjugation operator.

The charge conjugation operator is antisymmetric matrix, $C^T = -C$, and it acts on gamma matrices as

$$C\Gamma^\mu C^{-1} = -(\Gamma^\mu)^T. \quad (\text{A.1})$$

Now we have

$$(C\Gamma^\mu)^T = (\Gamma^\mu)^T C^T = -(\Gamma^\mu)^T C = C\Gamma^\mu C^{-1} C = C\Gamma^\mu, \quad (\text{A.2})$$

$$(C\Gamma^\mu \Gamma^\nu)^T = C\Gamma^\mu \Gamma^\nu \implies (C\Gamma^{[\mu\nu]})^T = C\Gamma^{[\mu\nu]}, \quad (\text{A.3})$$

$$(C\Gamma^\mu \Gamma^\nu \Gamma^\rho)^T = -C\Gamma^\mu \Gamma^\nu \Gamma^\rho \implies (C\Gamma^{[\mu\nu\rho]})^T = -C\Gamma^{[\mu\nu\rho]}, \quad (\text{A.4})$$

$$(C\Gamma^\mu \Gamma^\nu \Gamma^\rho \Gamma^\lambda)^T = -C\Gamma^\mu \Gamma^\nu \Gamma^\rho \Gamma^\lambda \implies (C\Gamma^{[\mu\nu\rho\lambda]})^T = -C\Gamma^{[\mu\nu\rho\lambda]}, \quad (\text{A.5})$$

$$(C\Gamma^\mu \Gamma^\nu \Gamma^\rho \Gamma^\lambda \Gamma^\omega)^T = C\Gamma^\mu \Gamma^\nu \Gamma^\rho \Gamma^\lambda \Gamma^\omega \implies (C\Gamma^{[\mu\nu\rho\lambda\omega]})^T = C\Gamma^{[\mu\nu\rho\lambda\omega]}. \quad (\text{A.6})$$

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